§2 Yang-Mills connections and anti-self-dual instantons

§2.1 Yang-Mills connecons

Let
- X : compact, oriented, smooth manifold. Fix a Riemannian metric g on X.
- G : compact Lie group
- $P \to X$, principal G-bundle over X

$\mathfrak{g}$: Lie algebra of G

$\langle\ ,\ \rangle$: invariant metric under the adjoint action of Lie group.

e.g. $\langle A, B\rangle = -\operatorname{tr}(AB)$ for $A, B \in \mathfrak{su}(2)$

$\Longrightarrow$ inner product $\langle\ ,\ \rangle$ on $\Lambda^k T^*X \otimes \mathfrak{g}_P$

Denote by $|\cdot|$ the norm induced by this.

<u>Yang-Mills functional</u> (or Y-M energy, Y-M action)

For $A \in \mathcal{A}_P := \{\text{all connections on } P\}$

define $\quad YM(A) := \int_X |F_A|^2 \, d\mathrm{vol}_g$.

F_A : the curvature of A.

As the metric is invariant under the adjoint action, and $F_{\varphi(A)} = \varphi^{-1} F_A \varphi$, where φ is a gauge transformation, this functional is <u>gauge invariant</u> i.e.

$YM(\varphi(A)) = YM(A)$

So, this is defined on $\mathcal{B}_P := \mathcal{A}_P / \mathcal{G}_P$.

<u>Yang-Mills connections</u>

We consider the Euler-Lagrange equation of Yang-Mills functional. i.e. critical points of Yang-Mills functional.

Take $A + t\alpha \in \mathcal{A}_P$, where $A \in \mathcal{A}_P$, $\alpha \in \Omega^1(X, \mathfrak{g}_P)$, $t \in \mathbb{R}$.

Then $\quad F_{A+t\alpha} = F_A + t\, d_A \alpha + \frac{1}{2} t^2 [\alpha \wedge \alpha]$,

So $\quad \left. \frac{d}{dt} YM(A+t\alpha) \right|_{t=0} = 2 \int_X \langle F_A, d_A \alpha \rangle \, d\mathrm{vol}_g$

Hence

A: a critical point of Yang-Mills functional $\iff d_A^* F_A = 0$

(d_A^*: the formal adjoint of d_A.)

Def.

A connection A is a Yang-Mills connection

if A satisfies $d_A^* F_A = 0$.

Remark. the Bianchi identity $d_A F_A = 0$.

So Yang-Mills connections can be seen as a non-linear analogue of harmonic forms ($d\alpha = 0$, $d^*\alpha = 0$)

§2.2 Anti-Self-dual instantons

X : compact, oriented smooth 4-manifold. with Riemann metric g.

$*_g$: the Hodge star operator

$$*_g : \Lambda^2_X \to \Lambda^2_X , \quad \Lambda^2_X := \Lambda^2 T^* X.$$

with $(*_g)^2 = 1$.

eigenvalues are ± 1.

$$\Rightarrow \Lambda^2_X = \Lambda^+_X \oplus \Lambda^-_X$$

$\Lambda^+_X|_x$: $+1$-eigen space of $*_g|_x$

$\Lambda^-_X|_x$: -1-eigen space of $*_g|_x$. $\quad x \in X$

This induces

$$\Omega^2(\mathfrak{g}_P) = \Omega^+(\mathfrak{g}_P) \oplus \Omega^-(\mathfrak{g}_P)$$

$$\Omega^\pm(\mathfrak{g}_P) = \Gamma(\Lambda^\pm_X \otimes \mathfrak{g}_P)$$

Write $F_A = F_A^+ + F_A^-$, $F_A^\pm \in \Omega^\pm(\mathfrak{g}_P)$

Def. A connection A is an anti-self-dual (ASD) connection (self-dual connection respectively) if A satisfies $F_A^+ = 0$ ($F_A^- = 0$)

Rmk. $F_A^+ = 0 \Longleftrightarrow *_g F_A = - F_A$

($F_A^- = 0 \Longleftrightarrow *_g F_A = F_A$)

Prop. ASD connections are Yang-Mills connections

$\because$)

$$d_A^* F_A = \pm *_g d_A *_g F_A$$

$$\underset{\text{ASD}}{=} \pm *_g d_A F_A .$$

$$\underset{\text{Bianchi}}{=} 0 \qquad \square$$

Rmk. the decomposition $\Lambda^2 = \Lambda^+_X \oplus \Lambda^-_X$ depends only on the conformal class of Riemannian metrics.

Chern-Weil theory

For simplicity, we take $G = SU(2)$

$E \to X$, associated vector bundle.

$$c_2(E).[X] = \frac{1}{8\pi^2}\int_X \mathrm{tr}(F_A^2) \in \mathbb{Z}.$$

the 2nd Chern class of E

fundamental class of X

Excercise

$$\mathrm{tr}(F_A^2) = -(|F_A^+|^2 - |F_A^-|^2)\,d\mathrm{vol}_g.$$

(Hint: $|F_A|^2 = -\mathrm{tr}(F_A \wedge *_g F_A)$)

So,

$$8\pi^2 c_2(E) = \int_X \mathrm{tr}(F_A^2)$$

$$= \int_X |F_A^-|^2\,d\mathrm{vol}_g - \int_X |F_A^+|^2\,d\mathrm{vol}_g$$

on the other hand,

$$YM(A) = \int_X |F_A|^2\,d\mathrm{vol}_g$$

$$= \int_X |F_A^+|^2\,d\mathrm{vol}_g + \int_X |F_A^-|^2\,d\mathrm{vol}_g$$

Thus

$$YM(A) = 2\int_X |F_A^+|^2 + 8\pi^2 c_2(E)$$

In particular,

$$YM(A) \geq 8\pi^2 c_2(E)$$

If $c_2(E) > 0$, then

$$A : \text{ASD} \iff YM(A) = 8\pi^2 c_2(E)$$

So ASD instantons attain the minimum of the Yang-Mills functional.

<u>ASD instanton moduli space</u>

$$\mathcal{M}_P := \{ A \in \mathcal{A}_P : F_A^+ = 0 \} / \mathcal{G}_P$$

Example: charge 1 ASD instanton moduli space on S^4.
(c_2 : the second Chern class)

(ref: M. Atiyah, " Geometry of Yang-Mills fields" (1979))

By the conformal invariance of ASD instantons and that of Yang-Mills functional in 4 dimensions

A : ASD on $\mathbb{R}^4$ with

$$\int_{\mathbb{R}^4} |F_A|^2 \, d\mathrm{vol}_g < \infty$$

$\Longleftrightarrow$ A : ASD on $S^4 \setminus \{\infty\}$ with

$$\int_{S^4} |F_A|^2 \, d\mathrm{vol}_g < \infty$$

$\Longrightarrow$ (removal singularity theorem by Uhlenbeck (Lect 6)) A is extended to a smooth ASD instanton on S^4.

we consider

$$\begin{cases} G = SU(2), \quad E \to S^4, & \text{associated } SU(2) \text{ vector bundle over } S^4. \\ c_2(E) = 1. & \end{cases}$$

ASD instantons on $\mathbb{R}^4 = \mathbb{H}$

$$x := x^1 + x^2 i + x^3 j + x^4 k \in \mathbb{H}.$$

$x_i \in \mathbb{R}$ $(i = 1, \cdots, 4)$

$\left(\begin{array}{l} i^2 = j^2 = k^2 = ijk = -1. \\ \text{and } i, j, k \text{ anti commute} \end{array} \right)$

$$\mathrm{Im}\, x = x^2 i + x^3 j + x^4 k.$$

$$\bar{x} = x^1 - x^2 i - x^3 j - x^4 k.$$

$$|x|^2 = x\bar{x}.$$

- anti-self-dual 2-form

$$d\bar{x} \wedge dx = 2 \left(\begin{array}{l} (dx^1 \wedge dx^2 - dx^3 \wedge dx^4) i \\ + (dx^1 \wedge dx^3 + dx^2 \wedge dx^4) j \\ + (dx^1 \wedge dx^4 - dx^2 \wedge dx^3) k \end{array} \right)$$

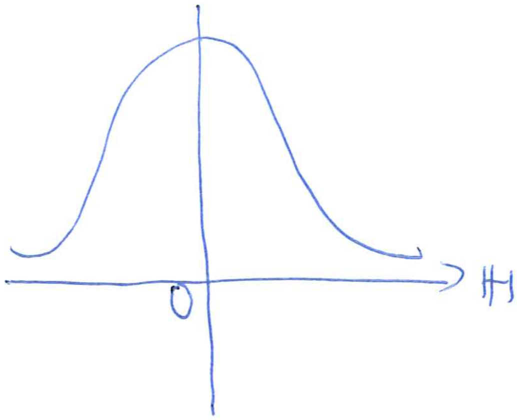
- self-dual 2-form

$$dx \wedge d\bar{x} = -2 \left(\begin{array}{l} (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) i \\ + (dx^1 \wedge dx^3 - dx^2 \wedge dx^4) j \\ + (dx^1 \wedge dx^4 + dx^2 \wedge dx^3) k \end{array} \right).$$

scale $\lambda > 0$, centre at 0 ASD instanton

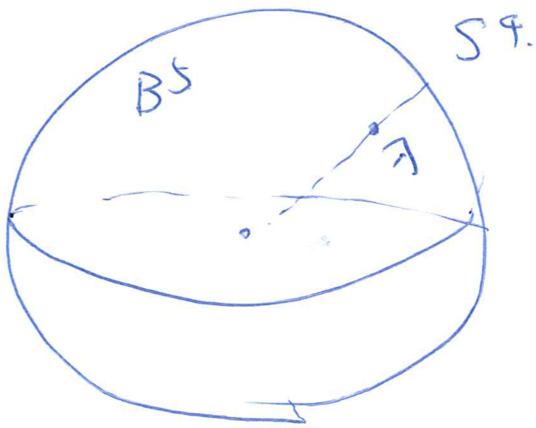
$$A_\lambda = \operatorname{Im}\left(\frac{x \, d\bar{x}}{\lambda^2 + |x|^2} \right)$$

the curvature $$F_{A_\lambda} = \frac{\lambda^2 \, dx \wedge d\bar{x}}{(\lambda^2 + |x|^2)}$$



<u>Fact</u>. all ASD instantons with $c_2(E) = 1$ on the $SU(2)$ vector bundle $E \to S^4$ are parametrized by the scales λ and the centres modulo gauge transformations

$$\Longrightarrow \quad \mathcal{M}_E = B^5$$



Gauge-theoretic moduli problem

$$\begin{array}{ll} \mathcal{E} & : \text{vector bundle, fibre} = \Omega^+_X(\mathfrak{g}_P) \\ \downarrow \ \uparrow & s : \text{section defined by } F_A^+ \\ \mathcal{B}_P := \mathcal{A}_P / \mathcal{G}_P & \end{array}$$

$$\left(\mathcal{E} := \mathcal{A}_P \times_{\mathcal{G}_P} \Omega^+_X(\mathfrak{g}_P) \right)$$

$$\mathcal{M}_P = s^{-1}(0)$$

Problem : construct a fundamental cycle out of $\mathcal{M}_P$

$\Rightarrow$ can integrate Chern classes of the "universal bundle" over $\mathcal{M}_P$ (intersection theory)

$\Rightarrow$ Donaldson invariants
(diffeomorphic invariants)

issues

- smoothness
- orientation
- compactness

§2.3 Higher-dimensional analogues of ASD instantons

Let X : compact, oriented, smooth manifold of $\dim_{\mathbb{R}} = n$

with Riemannian metric g.

Ω : closed $(n-4)$-form on X.

$E \to X$, vector bundle over X.

Def A connection on E is an Ω-ASD connection

if $*_g F_A = -F_A \wedge \Omega$

Prop Ω-ASD connections are Yang-Mills ones

Examples

- G_2-instantons on G_2-manifolds : Ω = associative 3-form
- Spin(7)-instantons on Spin(7)-manifolds : Ω = Cayley 4-form
- Hermitian Yang-Mills connections on Kähler manifolds : $\Omega = \dfrac{\omega^{m-2}}{(m-2)!}$

  $n = 2m$

  ω : Kähler form

- complex ASD connections on Calabi-Yau 4-folds : $\Omega = 4\,\mathrm{Re}(\theta) + \frac{1}{2}\omega^2$

θ : holomorphic 4-form.